

# SMALL PRODUCT SETS IN COMPACT GROUPS

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**ABSTRACT.** This paper addresses the structure of pairs  $(A, B)$  of Borel sets in a compact and second countable group  $G$  with Haar probability measure  $m_G$  which have positive  $m_G$ -measures and satisfy the identity

$$m_G(AB) = m_G(A) + m_G(B) < 1.$$

When  $G$  is a compact, *abelian and connected* group, M. Kneser provided a very satisfactory description of such pairs, which roughly says that both  $A$  and  $B$  must be co-null subsets of pre-images of two closed intervals under the same surjective homomorphism onto the circle group. Much more recently, Griesmer was able to divide the set of all possible such pairs in any compact and *abelian* group into four (not completely disjoint) subclasses.

Motivated by some applications to small product sets in *countable amenable groups* with respect to the left upper Banach density, we continue this study, and offer a description of pairs of Borel sets as above in any compact and second countable group with an *abelian* identity component. The relevance of these compact groups to the study of product sets in countable amenable groups will be explained. The main result of this paper asserts that in this *non-abelian* situation, essentially only one new class of examples of pairs as above can occur.

Unfortunately, due to a wide variety of technicalities we will have to spend a non-trivial part of the introduction to carefully set up the tools and concepts necessary for the formulation of the main results and to prepare for their proofs.

## 1. INTRODUCTION

**1.1. General comments.** Let  $G$  be a compact and second countable group with Haar probability measure  $m_G$ . Suppose  $(A, B)$  is a pair of Borel sets in  $G$  and define their *product set* by

$$AB = \{ab : a \in A, b \in B\}.$$

Although the product set of  $(A, B)$  may fail to be Borel measurable, it is always measurable with respect to the *completion* of the Borel  $\sigma$ -algebra relative to  $m_G$ , so in particular the Haar measure of the product set  $AB$  is always well-defined.

A well-studied and fundamental question in additive combinatorics is how "small" the  $m_G$ -measure of  $AB$  can be in terms of  $m_G(A)$  and  $m_G(B)$ . Since  $m_G$  is invariant under both left and right translations, it is clear that one always has the lower bound

$$m_G(AB) \geq \max(m_G(A), m_G(B))$$

and equality happens if and only if either  $A \subset B$  and  $B$  is a conull subset of an open subgroup of  $G$  or  $B \subset A$  and  $A$  is a conull subset of an open subgroup of  $G$ .

A less extreme example of a "small" product set would be when

$$m_G(AB) < \min(1, m_G(A) + m_G(B)). \tag{1.1}$$

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This kind of pairs have been extensively studied over the years, and are often referred to as *sub-critical* pairs, although sometimes the term *critical* is also used, which may cause some confusion when comparing the results in this paper with other results in the literature.

When  $G$  is *abelian*, Kneser [10] showed that if  $(A, B)$  is a sub-critical pair, then there exist a *finite* abelian group  $M$ , a homomorphism  $p : G \rightarrow M$  and subsets  $I, J \subset M$  such that  $m_G(AB) = m_M(IJ)$ , where  $m_M$  is the normalized counting measure (Haar measure) on  $M$ , and

$$A \subset p^{-1}(I) \quad \text{and} \quad B \subset p^{-1}(J)$$

and

$$m_M(IJ) = m_M(I) + m_M(J) - m_M(\{e\}), \quad (1.2)$$

where  $e$  denotes the identity element in  $M$ . In other words, the problem of understanding the structure of Borel sets  $(A, B)$  which satisfy (1.1) "reduces" to a large extent to the problem of understanding the structure of subsets  $(I, J)$  in a *finite* abelian group  $M$  which satisfy (1.2). In the case when  $G$  is finite cyclic group of prime order, the latter problem was solved by Vosper in [13]. The situation for general finite *abelian* groups was later outlined by Kemperman in [8].

When  $G$  is *not* necessarily abelian, Kemperman showed in [9] that Kneser's "reduction" essentially prevails, modulo the identity (1.2), that is to say, if  $(A, B)$  is a sub-critical pair in  $G$ , then there exists a *finite* group  $M$ , a homomorphism  $p : G \rightarrow M$  and subsets  $I, J \subset M$  with  $m_G(AB) = m_M(IJ)$  such that

$$A \subset p^{-1}(I) \quad \text{and} \quad B \subset p^{-1}(J),$$

but the identity (1.2) may fail. Recently, DeVos [2] was able to find a reasonable substitute for (1.2) and prove some partial results on the structure of the sets which satisfy this constraint.

Since compact and *connected* groups do not admit any non trivial homomorphisms onto finite groups, Kemperman's Theorem (and Kneser's Theorem in the case of abelian groups) in particular shows that if  $(A, B)$  is a pair of Borel sets in such a group, then

$$m_G(AB) \geq \min(1, m_G(A) + m_G(B)). \quad (1.3)$$

Clearly, if  $G$  is the circle group and  $A$  and  $B$  are co-null Borel sets of two closed intervals in  $G$ , then  $(A, B)$  satisfies

$$m_G(AB) = \min(1, m_G(A) + m_G(B)), \quad (1.4)$$

which shows that the inequality (1.3) is sharp in this case. We shall refer to Borel pairs  $(A, B)$  in a compact group  $G$  which satisfies (1.4) as *critical pairs* (but we again stress that this is not a completely standard notion, and the reader should be cautious when comparing the results in this paper with other results in the literature).

It is not hard to see that if  $G$  is a compact group which admits a surjective homomorphism  $p$  onto the circle group  $\mathbb{T}$  and  $(I, J)$  is a pair of closed intervals in  $\mathbb{T}$  and  $A$  and  $B$  are co-null subsets of  $p^{-1}(I)$  and  $p^{-1}(J)$  respectively, then  $(A, B)$  is a critical pair, i.e. it satisfies (1.4).

When  $G$  is a *compact and connected abelian group*, Kneser established a beautiful converse of this observation, namely that the existence of a pair of Borel sets in  $G$  which satisfies (1.4) *forces* the existence of a homomorphism  $p : G \rightarrow \mathbb{T}$  and closed intervals  $I, J \subset \mathbb{T}$  as above. The situation for general compact (not necessarily abelian) connected groups seems to be largely unknown.

In order to establish his theorem on critical pairs in compact and connected *abelian* groups, Kneser utilized a very useful tool in additive combinatorics known as the *Dyson e-transform*. This "transform" roughly works as follows. Let  $G$  be a compact *abelian* group and suppose  $(A, B)$  is a pair of Borel sets in  $G$  with positive  $m_G$ -measures. Set  $A_0 = A$  and  $B_0 = B$  and for every  $n \geq 1$ , we choose an element  $x_n \in A_{n-1}$  and define the Borel sets

$$A_n = A_{n-1} \cup B_{n-1}x_n \quad \text{and} \quad B_n = x_n^{-1}A_{n-1} \cap B_{n-1}.$$

Since  $G$  is abelian, we have

$$A_n \subset A_{n+1} \quad \text{and} \quad B_n \supset B_{n+1}$$

and

$$A_n B_n \subset AB \quad \text{and} \quad m_G(A_n) + m_G(B_n) = m_G(A) + m_G(B)$$

for all  $n$ . In particular, if  $(A, B)$  is a critical pair in  $G$ , then  $(A_n, B_n)$  is either a critical or sub-critical pair for every  $n$ , and upon studying the "limits" of these pairs, one can hope to deduce some structural information about the pair  $(A, B)$ . Kneser assumed connectedness of  $G$  in order to avoid some technical issues with open subgroups in the general setting. More recently, Griesmer in [5] (inspired by another recent paper [6] by Gryniewicz) was able to carry out the full analysis of critical pairs in compact *abelian* groups using the Dyson *e-transform* and to conclude that one could divide the possible critical pairs in any compact *abelian* into four (not disjoint) sub-classes.

According to this division, a critical pair  $(A, B)$  of Borel sets in  $G$  is either (up to some translations) of the form as in Kneser's Theorem, i.e. the sets  $A$  and  $B$  are conull subsets of pre-images of closed intervals under a homomorphism of an open subgroup of  $G$  to the circle group *or* there exists an open subgroup  $U < G$  and conull Borel sets  $A' \subset A$  and  $B' \subset B$  such that for some  $x, y \in G/U$ , the pair  $(x^{-1}A' \cap U, yB'^{-1} \cap U)$  is sub-critical in  $U$ . We shall refer to the latter situation by saying that  $(A, B)$  is a *locally sub-critical pair* in  $G$ . Griesmer [5] has been able to identify three (not mutually different) ways in which this local sub-criticality takes place. We shall not reproduce his findings here.

**1.2. A relativisation of Kneser's Theorem.** The aim of this paper is to understand the structure of critical Borel pairs in compact, but not necessarily *abelian* groups. Because of the possible non-abelianess of the groups under study, we cannot hope to successfully employ Dyson's *e-transform* as in Kneser's and Griesmer's approaches, so we have to develop an alternative route, which we shall now briefly outline.

Let  $G$  be a compact and second countable group and let  $N$  be a closed *normal* subgroup of  $G$ . We may view the Haar probability measure on  $N$  as the unique  $N$ -invariant probability measure supported on  $N < G$ . Similarly, given any element  $x \in G/N$ , the right translate of  $m_N$  by  $x$  is a  $N$ -invariant probability measure on the coset  $Nx$ , and we use the rather ugly symbol  $(m_G)_x^N$  to denote this probability measure. One readily checks that the map  $x \mapsto (m_G)_x^N$  from  $G/N$  to the space of Borel probability measures on  $G$  is weak\*-continuous and by the uniqueness of  $m_G$ , we have the following disintegration of  $m_G$ ,

$$m_G = \int_{G/N} (m_G)_x^N dm_{G/N}(x), \tag{1.5}$$

where  $m_{G/N}$  denotes the push-forward of  $m_G$  onto the quotient group  $G/N$ .

Suppose that  $(A, B)$  is a critical Borel pair in  $G$  and let  $N$  be a closed normal subgroup. For every  $x, y \in G$ , we can define the Borel sets

$$A_x = x^{-1}A \cap N \quad \text{and} \quad B_y = By^{-1} \cap N$$

in  $N$ . Note that since  $N$  is assumed to be normal, we have  $A_x B_y \subset x^{-1}(AB \cap Nxy)y^{-1}$  and

$$(m_G)_x^N(A) = m_N(x^{-1}A) \quad \text{and} \quad (m_G)_y^N(B) = m_N(By^{-1}).$$

We shall say that a Borel pair  $(A, B)$  is *critical with respect to*  $N$  if there exist conull Borel subsets  $X \subset G$  and  $Z \subset X \times X$  such that

$$(m_G)_x^N(A) = m_G(A) \quad \text{and} \quad (m_G)_y^N(B) = m_G(B)$$

for all  $x, y \in X$  and

$$(m_G)_{xy}^N(AB) = (m_G)_x^N(A) + (m_G)_y^N(B)$$

for all  $(x, y) \in Z$ .

Note that if  $N$  is *connected* and  $(A, B)$  is a critical pair with respect to  $N$ , then all of the pairs  $(A_x, B_y)$  are critical for  $(x, y) \in Z$ , that is to say, in this case we have a function from  $Z$  into the set of critical pairs in  $N$ . Hence, if  $N$  in addition is *abelian*, then Kneser's Theorem can be applied to each of these pairs, producing for every  $z$  in  $Z$  a surjective homomorphism  $\pi_z$  of  $N$  onto the circle group  $\mathbb{T}$  and two elements  $\alpha_z, \beta_z \in \mathbb{T}$  such that

$$A_x \subset \pi_z^{-1}(\alpha_z I) \quad \text{and} \quad B_y \subset \pi_z^{-1}(J\beta_z),$$

where  $I$  and  $J$  are the uniquely determined closed intervals in  $\mathbb{T}$  which are symmetric around zero and have normalized arc lengths (Haar measures) equal to  $m_G(A)$  and  $m_G(B)$  respectively.

A prominent class of examples of critical pairs with respect to a closed normal subgroup can be given as follows. Let  $G$  be a compact group which is a semidirect product of the form  $G = N \rtimes K$ , where  $N$  and  $K$  are closed subgroups and  $K$  acts on  $N$  by conjugation, and suppose that  $\pi$  is a continuous homomorphism onto either  $\mathbb{T}$  or the "twisted" torus  $\tilde{\mathbb{T}} = \mathbb{T} \rtimes \{-1, 1\}$ . In the first case, it is not hard to see that the pair  $(\pi^{-1}(I), \pi^{-1}(J))$  is critical with respect to  $N$  for *any* pair of closed intervals  $I, J \subset \mathbb{T}$ .

In the second case, we let  $I$  and  $J$  be closed and *symmetric* intervals in  $\mathbb{T}$  and define the closed subsets  $\tilde{I} = I \rtimes \{-1, 1\}$  and  $\tilde{J} = J \rtimes \{-1, 1\}$  of  $\tilde{\mathbb{T}}$ . One can check that  $(\tilde{I}, \tilde{J})$  is a critical pair in  $\tilde{\mathbb{T}}$  and that the pair

$$A = \pi^{-1}(\tilde{I}) \quad \text{and} \quad B = \pi^{-1}(\tilde{J})$$

is critical in  $G$  with respect to  $N$ . We shall refer to this class of examples as *sturmian pairs*.

The main novelty in this paper is the observation that if  $G$  is any compact and second countable group with a closed and connected subgroup  $N$  such that  $G$  can be written on the semi-direct form  $G = N \rtimes K$  (one often says that  $N$  *splits* in  $G$ ) and if  $(A, B)$  is a Borel pair in  $G$  which is critical with respect to  $N$ , then the map  $(x, y) \mapsto (A_x, B_y)$  into the set of critical pairs in  $N$  must essentially come from a sturmian pair.

Recall that the *identity component* of a compact group  $G$  is defined as the intersection of all *open* subgroups and is thus automatically normal (in fact, it is fully characteristic, i.e. invariant under all continuous automorphisms of  $G$ ). If  $G$  is also second countable, then the

identity component can be realized as the intersection of a *countable* number of open (normal) subgroups.

**Theorem 1.1.** *Let  $G$  be a compact and second countable group with a split identity component  $N$  and suppose  $(A, B)$  is a critical pair in  $G$  with respect to  $N$ . Let  $I$  and  $J$  denote the closed and symmetric intervals in  $\mathbb{T}$  whose Haar probability measures equal  $m_G(A)$  and  $m_G(B)$  respectively. If  $N$  is abelian, then either there exists a continuous homomorphism  $\pi_0 : G \rightarrow \mathbb{T}$  such that*

$$A \subset \pi_0^{-1}(s'I) \quad \text{and} \quad B \subset \pi_0^{-1}(Jt')$$

*for some  $s', t' \in \mathbb{T}$ , or there exists a continuous homomorphism  $\pi : G \rightarrow \mathbb{T} \rtimes \{-1, 1\}$  such that*

$$A \subset \pi^{-1}\left(s' (I \rtimes \{-1, 1\})\right) \quad \text{and} \quad B \subset \pi^{-1}\left((J \rtimes \{-1, 1\}) t'\right)$$

*for some  $s', t' \in \mathbb{T} \rtimes \{-1, 1\}$ .*

Unfortunately, there are (quite intricate) examples of compact and second countable groups whose identity components do *not* split as above into a semi-direct product, so Theorem 1.1 is not directly applicable to those groups. However, the following theorem of Lee [11] allows us to at least partially circumvent this technicality.

**Dong Hoon Lee's Theorem.** *Let  $G$  be a compact group and let  $N$  denote its identity component. Then there exists a closed totally disconnected subgroup  $K$  of  $G$  and a surjective homomorphism  $p : N \rtimes K \rightarrow G$  whose kernel is contained in the center of  $N \cap K$ .*

In other words, if  $(A, B)$  is a critical pair in a compact group  $G$  with identity component  $N$ , then  $(p^{-1}(A), p^{-1}(B))$  is a critical pair in the semi-direct product  $N \rtimes K$ , for some totally disconnected subgroup  $K$  of  $N$  and surjective homomorphism  $p : N \rtimes K \rightarrow G$ .

Theorem 1.1 settles the question about the structure of those critical pairs which are critical *with respect to an abelian identity component*. However, far from every critical pair in a compact (non-connected) group is critical with respect to its identity component.

The second main observation of this paper roughly says that if a critical pair in a compact and second countable group  $G$  is *not* critical with respect to the identity component of the group, then the pair must exhibit substantial "periodicity"; in fact, such a Borel pair  $(A, B)$  in  $G$  must be *locally sub-critical*, i.e. there exists an open normal subgroup  $U < G$  and  $x, y \in G$  such that the pair  $(x^{-1}A \cap U, y^{-1}B \cap U)$  is sub-critical in  $U$ . Formally, this "relativization" can be stated as follows.

**Relativization of critical pairs (Appendix I).** *Let  $G$  be a compact and second countable group and suppose  $(A, B)$  is a critical pair in  $G$  which is not locally sub-critical. Then there exist an open subgroup  $L < G$  and conull  $\sigma$ -compact sets*

$$A' \subset A \quad \text{and} \quad B' \subset B$$

*and  $s, t \in G$  such that*

$$sA' \subset L \quad \text{and} \quad B't \subset L$$

*and  $(sA', B't)$  is critical in  $L$  with respect to the identity component of  $L$ .*

This relativization principle, together with Theorem 1.1 and Dong-Hoon Lee's Theorem, shows that a critical pair  $(A, B)$  which are not sub-critical pairs, admits *conull* subsets  $A' \subset A$  and  $B' \subset B$  with a rather satisfactory structure theory, namely they are realized as conull subsets of sturmian pairs as above. However, a priori, the original sets  $A$  and  $B$  do not need to be contained in these sturmian pairs. In the next subsection, we shall discuss a quite general "rigidity" principle which utilizes the special topological structure of sturmian pairs to conclude that one indeed does have global containment in a sturmian pair.

**1.3. Reduction theory of critical pairs.** At this point we have seen how a series of operations can be applied to a critical pair in order to put it in a form so that Theorem 1.1 can be applied. It makes sense to introduce a concise notion which neatly summarizes these operations.

In order to motivate this notion, we first observe that we so far have used three different ways to construct "new" critical pairs from a given critical pair  $(A, B)$  in a compact group  $G$ . Firstly, note that if  $L$  is an open (not necessarily normal) subgroup of  $G$  and there exists  $s, t \in G$  such that

$$sA \subset L \quad \text{and} \quad Bt \subset L,$$

then  $(sA, Bt)$  is a critical pair in  $L$ . Secondly, if  $S$  is a compact group and  $p : S \rightarrow G$  is a continuous surjective homomorphism, then

$$(p^{-1}(A), p^{-1}(B))$$

is critical in  $S$ . Thirdly, if it so happens that there exists a compact subgroup  $K < \text{Aut}(G)$  which fixes the set  $B$ , then

$$(A \rtimes K, B \rtimes K)$$

is a critical pair in the semi-direct product  $S = G \rtimes K$ . Upon combining these three ways to construct critical pairs, we arrive at the following "reduction theory".

Given two Borel sets  $A, A' \subset G$ , we shall say that  $A$  is *almost contained* in  $A'$  and denote this by  $A \subseteq A'$  if the relative complement  $A \setminus A'$  is a null set in  $G$ .

**Definition 1.1.** Let  $G$  and  $M$  be compact groups and suppose  $(A, B)$  and  $(I, J)$  are two pairs of Borel sets of  $G$  and  $M$  respectively. We shall say that  $(A, B)$  *almost reduces* to  $(I, J)$ , and write

$$(A, B) \preceq_a (I, J),$$

if the identity  $m_G(AB) = m_M(IJ)$  holds and there exist an open (not necessarily normal) subgroup  $L < G$  and  $s, t \in G$  such that  $sA \subset L$  and  $Bt \subset L$  and a compact group  $S$ , a continuous homomorphism  $p : S \rightarrow L$  and a compact (possibly trivial) subgroup  $K < \text{Aut}(M)$ , which fixes the set  $J$  such that

$$p^{-1}(sA) \subseteq \pi^{-1}(s'(I \rtimes K)) \quad \text{and} \quad p^{-1}(Bt) \subseteq \pi^{-1}((J \rtimes K)t')$$

for some  $s', t' \in M \rtimes K$ . We shall say that  $(A, B)$  *reduces* to  $(I, J)$ , and write

$$(A, B) \preceq (I, J),$$

if the almost containments above are inclusions, and we say that  $(A, B)$  *strictly reduces* to  $(I, J)$  if we have

$$G = S = L$$

that is to say, if no passage to neither subgroups nor supergroups is necessary.

**1.4. Rigidity of critical pairs.** From the discussion in the last subsection, we observe that Theorem 1.1, together with Dong-Hoon Lee's Theorem and the Relativization Principle, implies that if  $G$  is a compact and second countable group with an abelian identity component, and  $(A, B)$  is a critical pair in  $G$  which is not locally sub-critical, then  $(A, B)$  *almost reduces* to a pair of closed and symmetric intervals  $(I, J)$  in  $M = \mathbb{T}$ . We now wish to show that because of the special topological structure of closed intervals in  $\mathbb{T}$ , namely that they are equal to the closures of their interiors and the identities

$$I = \bigcap_{j \in J} Ij^{-1} \quad \text{and} \quad J = \bigcap_{i \in I} i^{-1}Ij \quad (1.6)$$

hold, the almost reduction is in fact a reduction.

The general framework for this "rigidity" of critical pairs can be described as follows. Let  $M$  be a compact group. We say that a Borel set  $I \subset M$  is *regular* if it equals the closure of its interior, and we say that a pair  $(I, J)$  of Borel sets in  $M$  is *left stable* if the inclusion

$$xJ \subset IJ$$

implies that  $x \in I$ . There is an obvious analogous notion of right stability, and we say that  $(I, J)$  is *stable* if it is both left and right stable. One readily checks that pairs of intervals in  $\mathbb{T}$  are stable, that is, the identities in (1.6) hold, and clearly closed intervals in  $\mathbb{T}$  are regular.

**Rigidity of critical pairs (Appendix II).** *Let  $G$  and  $M$  be compact groups and suppose  $(A, B)$  and  $(I, J)$  are critical pairs in  $G$  and  $M$  respectively. If  $(I, J)$  is stable and regular and if  $(A, B)$  almost reduces to  $(I, J)$ , then  $(A, B)$  in fact reduces to  $(I, J)$ .*

We can now neatly summarize the above discussions in the following theorem, which is a structural result for arbitrary critical pairs in any compact and second countable group with an abelian identity component.

**Theorem 1.2.** *Let  $G$  be a compact and second countable group with an abelian identity component. If  $(A, B)$  is a critical pair in  $G$ , which is not locally sub-critical, then  $(A, B)$  reduces to a sturmian pair.*

In Appendix III, we shall discuss some applications of Theorem 1.1 and Theorem 1.2 to problems about product sets in countable amenable groups, and hopefully also clarify the relevance of compact groups with abelian identity components in this setting.

**Remark.** It is not hard to show that if  $G$  is abelian and  $(A, B)$  is a critical pair in  $G$ , then we can in fact find an open subgroup  $L < G$ , a continuous surjective homomorphism  $\theta : L \rightarrow \mathbb{T}$  and closed intervals  $I, J \subset \mathbb{T}$  with

$$m_L(A) = m_{\mathbb{T}}(I) \quad \text{and} \quad m_L(B) = m_{\mathbb{T}}(J)$$

such that  $sA \subset L$  and  $Bt \subset L$  and

$$sA \subset \theta^{-1}(I) \quad \text{and} \quad Bt \subset \theta^{-1}(J)$$

for some  $s, t \in G$ , that is to say, in the case when  $G$  is abelian, the passage from  $L$  to a supergroup  $S$  in the definition of a reduced pair is unnecessary.

**1.5. An outline of Theorem 1.1.** The aim of this section is to reduce the proof of Theorem 1.1 to two propositions, of which the first one constitutes the technical core of this paper and the second one follows rather easily from standard arguments.

We shall say that a compact and connected group  $N$  is a Kneser group if every critical pair in  $N$  *strictly reduces* to a sturmian pair, and if  $G$  is a compact group with identity component  $N$ , then we say that  $N$  *splits* if there exists a totally disconnected subgroup  $Q < G$  such that  $G$  is isomorphic (as a topological group) to the semi-direct product  $G = N \rtimes Q$ .

**Proposition 1.1.** *Let  $G$  be a compact and second countable group with a split Kneser identity component  $N$ . If  $(A, B)$  is a critical pair in  $G$  with respect to  $N$ , then there exist a continuous surjective homomorphism  $\xi : N \rightarrow \mathbb{T}$  and closed symmetric intervals  $I_o, J_o \subset \mathbb{T}$  with*

$$m_G(A) = m_{\mathbb{T}}(I_o) \quad \text{and} \quad m_G(B) = m_{\mathbb{T}}(J_o)$$

*and conull Borel sets  $X \subset G$  and  $Y \subset X \times X$  and two Borel measurable maps  $\alpha_o, \beta_o : X \rightarrow \mathbb{T}$  such that*

$$p^{-1}A \cap N \subset \xi^{-1}(I_o \alpha_o(p)) \quad \text{and} \quad Bq^{-1} \cap N \subset \xi^{-1}(J_o \beta_o(q))$$

*for all  $(p, q) \in Y$ . Furthermore, there exists a continuous homomorphism  $\epsilon : G/N \rightarrow \{-1, 1\}$  such that*

$$\xi(c_p(n)) = \xi(n)^{\epsilon(n)}, \quad \forall (n, p) \in N \times G/N$$

*and*

$$(\alpha_o(p)\beta_o(q))^{\epsilon(p)} = (\alpha_o(p')\beta_o(q'))^{\epsilon(p')},$$

*whenever  $(p, q)$  and  $(p', q')$  belong to  $Y$  and satisfy  $pq = p'q'$ , where  $c_p$  denotes the automorphism of  $N$  induced by  $p$  in  $G$ .*

The strange-looking identities at the end of the formulation of Proposition 1.1 will be used to build Borel measurable maps from a conull Borel subset  $X \subset G$  to either the one-dimensional torus  $\mathbb{T}$  or the "twisted" torus  $\mathbb{T} \rtimes \{-1, 1\}$  with the property that there exist a conull Borel set  $Z \subset X \times X$  such that

$$\alpha(x)\beta(y) = \alpha(x')\beta(y')$$

for all  $(x, y) \in Z$  with  $xy = x'y'$ . The following proposition (which follows from rather standard arguments) shows that this kind of maps must come from translations of continuous homomorphisms between the relevant groups.

**Proposition 1.2.** *Let  $G$  and  $M$  be compact and second countable groups and suppose there exist conull Borel sets  $X \subset G$  and  $Z \subset X \times X$  and two Borel measurable maps  $\alpha, \beta : X \rightarrow M$  such that*

$$\alpha(x)\beta(y) = \alpha(x')\beta(y')$$

*whenever  $(x, y)$  and  $(x', y')$  belong to  $Z$  and satisfy  $xy = x'y'$ . Then there exists a continuous homomorphism  $\pi : G \rightarrow M$  and  $s, t \in M$  such that*

$$\alpha(x) = s\pi(x) \quad \text{and} \quad \beta(y) = \pi(y)t$$

*for almost every  $(x, y)$ .*



*Proof.* By Fubini's Theorem, there exist  $x_1, x_2 \in X$  such that the sets

$$Z^{x_1} = \{x \in G : (x_1, x) \in Z\} \quad \text{and} \quad Z_{x_2} = \{x \in G : (x, x_2) \in Z\}$$

are conull Borel sets in  $G$ . Note that

$$X' = x_1^{-1} X \cap X x_2^{-1} \quad \text{and} \quad Z' = (x_1, e)^{-1} Z (e, x_2)^{-1} \cap (X_o \times X_o)$$

are again conull Borel sets in  $G$  and  $G \times G$  respectively and

$$Z'^e = Z^{x_1} x_2^{-1} \quad \text{and} \quad Z'_e = x_1^{-1} Z_{x_2}.$$

If we set

$$X'' = Z'^e \cap Z'_e \quad \text{and} \quad Z'' = Z' \cap p^{-1}(X'') \cap (X'' \times X''),$$

where  $p : G \times G \rightarrow G$  denotes the product map, then  $X''$  and  $Z''$  are conull subsets,

$$e \in X'' \quad \text{and} \quad (e, e) \in Z''$$

and the functions

$$\gamma_1(x) = \alpha(x_1)^{-1} \alpha(x_1 x) \quad \text{and} \quad \gamma_2(x) = \beta(x x_2) \beta(x_2)^{-1}$$

are Borel measurable on  $X''$  and satisfy  $\gamma_1(e) = \gamma_2(e) = e$  and

$$\gamma_1(x) \gamma_2(y) = \gamma_1(x') \gamma_2(y')$$

whenever  $(x, y)$  and  $(x', y')$  belong to  $Z''$  and  $xy = x'y'$ . In particular,  $\gamma_1(x) = \gamma_2(x)$  for all  $x \in X''$  and the equation

$$\gamma_1(xy) = \gamma_1(x) \gamma_1(y)$$

holds whenever  $(x, y)$  belongs to  $Z''$ .

By Theorem B.2 in [14], we conclude that here exists a continuous homomorphism  $\pi : G \rightarrow M$  such that  $\pi(x) = \gamma_1(x)$  almost everywhere, and if we define

$$s = \alpha(x_1) \quad \text{and} \quad t = \beta(x_2),$$

then

$$\alpha(x) = s \pi(x) \quad \text{and} \quad \beta(x) = \pi(x) t$$

for almost every  $x$  in  $G$ . □

**1.5.1. Proof of Theorem 1.1.** Let  $G$  be a compact and second countable group with a split identity component which we shall assume is a Kneser group, i.e. we write

$$G = N \rtimes Q,$$

where  $Q < \text{Aut}(N)$  and every critical pair in  $N$  strictly reduces to a sturmian pair. Denote by  $c_p$  the automorphism of  $N$  given by  $p$  in  $Q$  and note that the map  $p \mapsto c_p$  is assumed to be a continuous homomorphism.

The group multiplication in  $G$  is given by

$$(m, p)(n, q) = (m c_p(n), pq), \quad \forall (m, p), (n, q) \in G.$$

In particular, the equation

$$(m, p)(n, q) = (m', p')(n', q')$$

holds if and only if

$$m c_p(n) = m' c_{p'}(n) \quad \text{and} \quad pq = p'q'.$$

Suppose  $(A, B)$  is a critical pair in  $G$  with respect to  $N$ . By Proposition 1.1, there exist a continuous surjective homomorphism  $\xi : G \rightarrow \mathbb{T}$ , closed symmetric intervals  $I_o, J_o \subset \mathbb{T}$  with

$$m_G(A) = m_{\mathbb{T}}(I_o) \quad \text{and} \quad m_G(B) = m_{\mathbb{T}}(J_o)$$

and conull Borel sets  $X \subset G$  and  $Y \subset X \times X$  and two Borel measurable maps  $\alpha_o, \beta_o : X \rightarrow \mathbb{T}$  with the property that

$$p^{-1}A \cap N \subset \xi^{-1}(I_o \alpha_o(p)) \quad \text{and} \quad Bq^{-1} \cap N \subset \xi^{-1}(J_o \beta_o(q))$$

for all  $p, q \in X$  and a continuous homomorphism  $\epsilon : Q \rightarrow \{-1, 1\}$  such that

$$\xi(c_p(n)) = \xi(n)^{\epsilon(n)}, \quad \forall (n, p) \in G$$

and

$$\left( \alpha_o(p) \beta_o(q) \right)^{\epsilon(p)} = \left( \alpha_o(p') \beta_o(q') \right)^{\epsilon(p')},$$

whenever  $(p, q)$  and  $(p', q')$  belong to  $Y$  and satisfy  $pq = p'q'$ .

First assume that  $\epsilon$  is trivial and define the maps  $\alpha, \beta : N \rtimes X \rightarrow \mathbb{T}$  by

$$\alpha(m, p) = \alpha_o(p)^{-1} \xi(m) \quad \text{and} \quad \beta(n, q) = \beta_o(q)^{-1} \xi(n)$$

and note that if  $(m, p), (n, q) \in N \rtimes X$  satisfy

$$(m, p)(n, q) = (m', p')(n', q'),$$

then

$$\begin{aligned} \alpha(m, p) \beta(n, q) &= (\alpha_o(p) \beta_o(q))^{-1} \xi(m) \xi(n) \\ &= (\alpha_o(p) \beta_o(q))^{-1} \xi(c_p(m)) \xi(n) \\ &= (\alpha_o(p) \beta_o(q))^{-1} \xi(c_p(m) n) \\ &= (\alpha_o(p') \beta_o(q'))^{-1} \xi(c_{p'}(m') n') \\ &= \alpha(m', p') \beta(n', q'). \end{aligned}$$

Furthermore, we have

$$A \cap (N \rtimes X) \subset \alpha^{-1}(I_o) \quad \text{and} \quad B \cap (N \rtimes X) \subset \beta^{-1}(J_o)$$

By Proposition 1.2, there exists a continuous homomorphism such that

$$\alpha(m, p) = s \pi(m, p) \quad \text{and} \quad \beta(n, q) = \pi(n, q) t$$

for some  $s, t \in G$  and for almost every  $(m, p)$  and  $(n, q)$  in  $G$ , which implies that

$$A \cap (N \rtimes X) \subset \pi^{-1}(s' I_o) \quad \text{and} \quad B \cap (N \rtimes X) \subset \pi^{-1}(J_o t')$$

for some  $s', t' \in \mathbb{T}$ . In particular, this shows that  $(A, B)$  almost reduces to  $(I_o, J_o)$ , but since  $(I_o, J_o)$  clearly is stable and both  $I_o$  and  $J_o$  are regular sets, the rigidity of critical pairs asserts that in fact  $A$  and  $B$  are contained in the right hand sides above.

Now assume that  $\epsilon$  is *not* trivial, which automatically means that it maps onto  $\{-1, 1\}$ . Define the maps

$$\alpha, \beta : N \rtimes X \rightarrow \mathbb{T} \rtimes \{-1, 1\}$$

by

$$\alpha(m, p) = \left( \alpha_o(p)^{-\epsilon(p)} \xi(m), \epsilon(p) \right) \quad \text{and} \quad \beta(n, q) = \left( \beta_o(q)^{-1} \xi(n), \epsilon(q) \right).$$

Note that if  $(m, p), (n, q) \in N \rtimes X$  satisfy

$$(m, p)(n, q) = (m', p')(n', q'),$$

then

$$\begin{aligned} \alpha(m, p) \beta(n, q) &= \left( (\alpha_o(p) \beta_o(q))^{-\epsilon(p)} \xi(m) \xi(n)^{\epsilon(p)}, \epsilon(p) \epsilon(q) \right) \\ &= \left( (\alpha_o(p) \beta_o(q))^{-\epsilon(p)} \xi(m c_p(n)), \epsilon(pq) \right) \\ &= \left( (\alpha_o(p') \beta_o(q'))^{-\epsilon(p')} \xi(m c_{p'}(n')), \epsilon(p'q') \right) \\ &= \alpha(m', p') \beta(n', q'). \end{aligned}$$

By Proposition 1.2, this shows that  $\alpha$  and  $\beta$  are indeed left and right translates of a continuous homomorphism  $\pi : G \rightarrow \mathbb{T} \rtimes \{-1, 1\}$  respectively. Since

$$\begin{aligned} \alpha^{-1}(I_o \rtimes \{-1, 1\}) \cap Np &= \left\{ (m, p) : \alpha_o(p)^{-\epsilon(p)} \xi(m) \in I_o \right\} \cap Np \\ &= \xi^{-1}(I_o \alpha_o(p)^{\epsilon(p)}) p \\ &= \xi^{-1}((I_o \alpha_o(p))^{\epsilon(p)}) p \\ &= (\xi \circ c_{p^{-1}})^{-1}(I_o \alpha_o(p)) p \\ &\supset A \cap Np \end{aligned}$$

and

$$\begin{aligned} \beta^{-1}(J_o \rtimes \{-1, 1\}) \cap Nq &= \left\{ (m, q) : \beta_o(q)^{-1} \xi(m) \in J_o \right\} \cap Nq \\ &= \xi^{-1}(J_o \beta_o(q)) q \\ &\supset B \cap Nq, \end{aligned}$$

we see that

$$A \cap (N \rtimes X) \subset \alpha^{-1}(I \rtimes \{-1, 1\}) \quad \text{and} \quad B \cap (N \rtimes X) \subset \beta^{-1}(J \rtimes \{-1, 1\}),$$

so we conclude as before (using the rigidity of critical pairs) that

$$A \subset \pi^{-1}(s(I \rtimes \{-1, 1\})) \quad \text{and} \quad B \subset \pi^{-1}(t(J \rtimes \{-1, 1\})),$$

which finishes the proof.

**1.6. Organization of the paper.** The paper consists of Section 2 and two appendices. In Section 2 we give a proof of Proposition 1.1. In Appendix I we establish the principle of relativization of critical pairs which was formulated in the introduction and used in the passage from Theorem 1.1 to Corollary 1.2. In Appendix II we prove the rigidity of critical pairs which was formulated in the introduction and used in the last step of the proof of Theorem 1.1.

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## 2. PROOF OF PROPOSITION 1.1

The aim of this section is to establish Proposition 1.1. This will be done by an iterative use of three lemmata which we shall prove below.

The first lemma asserts that two sets in a compact group  $G$  which are pre-images of nice enough sets in a compact group  $M$  under two surjective homomorphisms from  $G$  onto  $M$  and only differ up to null sets must in fact be equal and the corresponding sets must be related via an automorphism of  $M$ .

If  $M$  is a compact group and  $I \subset M$  is a subset, then the *stabilizer group*  $M_I$  of  $I$  is defined by

$$M_I = \{m \in M : mI = Im = I\}.$$

Recall that a set  $I \subset M$  is *regular* if it equals the closure of its interior in  $M$ .

**Lemma 2.1.** *Let  $G$  and  $H$  be compact and second countable groups and suppose  $\pi_1$  and  $\pi_2$  are continuous surjective homomorphisms from  $G$  onto  $H$ . If  $I_1, I_2 \subset H$  are regular sets with trivial stabilizer groups such that*

$$\pi_1^{-1}(I_1) \sim \pi_2^{-1}(I_2),$$

*then there exists a continuous automorphism  $\alpha$  of  $H$  such that*

$$\pi_1 = \alpha \circ \pi_2 \quad \text{and} \quad I_1 = \alpha(I_2).$$

*In particular, if  $H = \mathbb{T}$  and  $I_1$  and  $I_2$  are closed intervals, then either  $\pi_1 = \pi_2$  and  $I_1 = I_2$ , or*

$$\pi_1(x) = \pi_2(x)^{-1} \quad \text{and} \quad I_1 = I_2^{-1}$$

*for all  $x$  in  $G$ .*

*Proof.* Since continuous *surjective* homomorphisms between  $G$  and  $H$  are necessarily open maps, we have

$$\pi_1^{-1}(I_1) = \overline{\pi_1^{-1}(I_1^o)} \quad \text{and} \quad \pi_2^{-1}(I_2) = \overline{\pi_2^{-1}(I_2^o)},$$

i.e. the pre-images of  $I_1$  and  $I_2$  under  $\pi_1$  and  $\pi_2$  are regular sets in  $G$ . By assumption, the open sets defined by

$$U_1 = \pi_1^{-1}(I_1^o) \setminus \pi_2^{-1}(I_2) \quad \text{and} \quad U_2 = \pi_2^{-1}(I_2^o) \setminus \pi_2^{-1}(I_1)$$

are Haar null and thus empty, which implies that

$$\pi_1^{-1}(I_1) = \overline{\pi_1^{-1}(I_1^o)} \subset \pi_2^{-1}(I_2) \quad \text{and} \quad \pi_2^{-1}(I_2) = \overline{\pi_2^{-1}(I_2^o)} \subset \pi_1^{-1}(I_1).$$

Hence,  $\pi_1^{-1}(I_1) = \pi_2^{-1}(I_2)$ , and since  $I_1$  and  $I_2$  are assumed to have trivial stabilizer subgroups, we conclude that  $\ker \pi_1 = \ker \pi_2$ . Indeed, if  $x \in \ker \pi_1$ , then

$$\pi_2(\pi_1^{-1}(I_1)) = I_2 = \pi_2(\pi_1^{-1}(I_1)) \pi_2(x)$$

which shows that  $\pi_2(x)$  fixes  $I_2$  and must thus be zero. Hence,  $\ker \pi_2 \subset \ker \pi_1$ , and by symmetry we conclude that  $\ker \pi_1 = \ker \pi_2$ .

If we denote by  $\pi$  the canonical quotient map from  $G$  onto the quotient group  $H = G/\ker \pi_1$ , then, by the universal property of this map, there exist continuous automorphisms  $\alpha_1$  and  $\alpha_2$  of  $H$  such that

$$\pi_1 = \alpha_1 \circ \pi \quad \text{and} \quad \pi_2 = \alpha_2 \circ \pi.$$

In particular, if we define  $\alpha = \alpha_1 \circ \alpha_2^{-1}$ , then  $\pi_1 = \alpha \circ \pi_2$ , which finishes the proof. The last assertion immediately follows from the fact that  $\text{Aut}(\mathbb{T}) = \{-1, 1\}$ .  $\square$

Before we state the second lemma, we first note that if  $N$  is a connected group and if  $\xi : N \rightarrow \mathbb{T}$  is a non-trivial homomorphism (hence surjective) then

$$\check{\xi}(n) = \xi(n)^{-1}, \quad n \in N,$$

is again a non-trivial homomorphism from  $N$  onto  $\mathbb{T}$ .

**Lemma 2.2.** *Let  $N$  be a connected group and denote by  $X$  the (possibly empty) set of all non-trivial continuous homomorphisms from  $N$  to  $\mathbb{T}$ . Then there exists a subset  $S \subset X$  such that*

$$X = S \cup \check{S} \quad \text{and} \quad S \cap \check{S} = \emptyset,$$

where

$$\check{S} = \{\xi \in X : \check{\xi} \in S\}.$$

Furthermore, if  $N$  in addition is second countable, then for any set  $S$  as above, there exists a Borel measurable map  $\epsilon : \text{Aut}(N) \rightarrow \{-1, 1\}$  such that

$$\epsilon(\alpha) \circ \xi \circ \alpha \in S$$

for all  $\alpha \in \text{Aut}(N)$ .

*Proof.* For the first assertion, it suffices to show that the self-map  $\xi \mapsto \check{\xi}$  does not admit a fixed point in  $X$ . However, the equation  $\check{\xi} = \xi$  holds if and only if the image of  $\xi$  is contained in the finite subgroup of order two elements of  $\mathbb{T}$ , which is impossible since  $N$  is connected.

For the second assertion, we note that since  $N$  is second countable, the Hilbert space  $L^2(N, m_N)$  is separable and since every two distinct element in  $X$  are orthogonal in  $L^2(N, m_N)$ , we see that  $X$  (and thus  $S$ ) is at most countable. In particular, the set

$$E = \{\alpha \in \text{Aut}(N) : \xi \circ \alpha \in S\}$$

is  $F_\sigma$  (so in particular Borel measurable), and thus the map defined by  $\epsilon(\alpha) = 1$  if  $\alpha \in E$  and  $\epsilon(\alpha) = -1$  if  $\alpha \notin E$  is Borel measurable and satisfies  $\epsilon(\alpha) \circ \xi \circ \alpha \in S$  for all  $\alpha$  in  $\text{Aut}(N)$ .  $\square$

The third and final lemma before we embark on the proof of Proposition 1.1 is technical but quite important. It will be used to show that certain maps which we construct, without any a priori regularity whatsoever, must in fact be Borel measurable.

**Lemma 2.3.** *Let  $G$  and  $M$  be compact groups and let  $N$  be a closed subgroup of  $G$ . Let  $C \subset G$  and  $I \subset M$  be Borel sets and suppose that there exist a continuous surjective homomorphism  $\xi : N \rightarrow M$ , a conull Borel set  $X \subset G$  and a map  $\gamma : X \rightarrow M$  such that*

$$x^{-1}C \cap N \sim \xi^{-1}(I\gamma(x))$$

for all  $x \in X$ . If  $M$  is second countable and the essential stabilizer group of  $I \subset M$  is trivial, then  $\gamma$  is Borel measurable on  $X$ .

*Proof.* Fix a countable basis  $(U_n)$  for the topology on  $M$  and note that

$$x^{-1}C \cap N \cap \xi^{-1}(U_n) \sim \xi^{-1}(I\gamma(x) \cap U_n)$$

for all  $n$  and for all  $x$  in  $X$ . Define the maps

$$\Psi : M \rightarrow [0, 1]^{\mathbb{N}} \quad \text{and} \quad \Phi : M \rightarrow [0, 1]^{\mathbb{N}}$$

by

$$\Psi(t)_n = m_M(I t \cap U_n) \quad \text{and} \quad \Phi(x)_n = m_N(x^{-1}C \cap N \cap \xi^{-1}(U_n))$$

for  $t \in M$  and  $x \in G$  and  $n$  in  $\mathbb{N}$ .

Clearly, both  $\Psi$  and  $\Phi$  are Borel measurable with respect to the product Borel structure on  $[0, 1]^{\mathbb{N}}$  and by construction, we have

$$\Psi(\gamma(x)) = \Phi(x), \quad \forall x \in G.$$

Since  $C$  has trivial essential stabilizer, Corollary 3.7 in Appendix I of this paper asserts that  $\Psi$  is injective, and is thus a Borel isomorphism onto its image (see e.g. Theorem A.4 in [14]). We conclude that the composition  $\gamma = \Psi^{-1} \circ \Phi$  is Borel measurable on  $X$ .  $\square$

**2.1. Proof of Proposition 1.1.** Let  $G$  be a compact and second countable group with a split identity component  $N$ , i.e. the group  $G$  can be written as a semi-direct product

$$G = N \rtimes Q,$$

where  $Q < \text{Aut}(N)$  is a closed subgroup isomorphic to the quotient group  $G/N$ . In particular,

$$m_G(C) = \int_Q m_N(p^{-1}C \cap N) dm_Q(p) = \int_Q m_N(Cq^{-1} \cap N) dm_Q(q)$$

for every Borel set  $C \subset G$ , where  $m_N$  and  $m_Q$  denote the Haar probability measures on  $N$  and  $Q$  respectively.

Suppose that  $(A, B)$  is a critical pair in  $G$  with respect to  $N$ . Recall that this means that there exists a conull subset  $Z \subset Q \times Q$  such that

$$(m_G)_{pq}^N(AB) = (m_G)_p^N(A) + (m_G)_q^N(B) < 1$$

and

$$(m_G)_p^N(A) = m_G(A) \quad \text{and} \quad (m_G)_q^N(B) = m_G(B)$$

for all  $(p, q)$  in  $Z$ . In our setting, these identities imply that

$$m_N((p^{-1}A \cap N)(Bq^{-1} \cap N)) \leq (m_G)_{pq}^N(AB) = m_N(p^{-1}ABq^{-1} \cap N)$$

and

$$(m_G)_p^N(A) + (m_G)_q^N(B) = m_N(p^{-1}A \cap N) + m_N(Bq^{-1} \cap N),$$

and thus

$$(p^{-1}A \cap N)(Bq^{-1} \cap N) \sim p^{-1}ABq^{-1} \cap N$$

and

$$m_N((p^{-1}A \cap N)(Bq^{-1} \cap N)) = m_N(p^{-1}A \cap N) + m_N(Bq^{-1} \cap N)$$

since  $N$  is connected, and thus cannot admit sub-critical pairs by Kneser's Theorem.

In particular, we see that

$$(p^{-1}A \cap N, B q^{-1} \cap N)$$

is a critical pair in  $N$  and

$$m_N(p^{-1}A \cap N) = m_G(A) \quad \text{and} \quad m_N(B q^{-1} \cap N) = m_G(B)$$

for all  $(p, q) \in Z$ . Let  $I_o$  and  $J_o$  denote the (uniquely defined) closed and symmetric intervals in  $\mathbb{T}$  with Haar measures equal to  $m_G(A)$  and  $m_G(B)$  respectively, and let  $\Sigma$  denote the set of non-trivial homomorphisms from  $N$  into  $\mathbb{T}$ .

Since  $N$  is a Kneser group, there exist maps (with no a priori Borel regularity whatsoever)

$$\xi : Z \rightarrow \Sigma \quad \text{and} \quad \alpha, \beta : Z \rightarrow \mathbb{T}$$

such that

$$p^{-1}A \cap N \subset \xi_z^{-1}(I_o \alpha(z)) \quad \text{and} \quad B q^{-1} \cap N \subset \xi_z^{-1}(J_o \beta(z))$$

and

$$p^{-1}ABq^{-1} \cap N \sim \xi_z^{-1}(I_o J_o \alpha(z) \beta(z)).$$

**Step 1** (Producing Borel regularity). By the first part of Lemma 2.2 there exists a set  $S \subset \Sigma$  such that

$$\Sigma = S \sqcup \check{S}.$$

Since

$$\xi^{-1}(I) = \check{\xi}^{-1}(I^{-1}), \quad \forall \xi \in \Sigma$$

and for every set  $I \subset \mathbb{T}$ , we may without loss of generality (upon possible modifying  $\alpha$  and  $\beta$ ) assume that  $\xi_z \in S$  for all  $z \in Z$ .

Note that

$$\xi_z^{-1}(I_o \alpha(z)) \sim \xi_{z'}^{-1}(I_o \alpha(z'))$$

whenever  $z = (p, q)$  and  $z' = (p, q')$  are contained in  $Z$  and

$$\xi_z^{-1}(J_o \beta(z)) \sim \xi_{z'}^{-1}(J_o \beta(z'))$$

whenever  $z = (p, q)$  and  $z' = (p', q)$  are contained in  $Z$ .

Since  $\xi_z$  and  $\xi_{z'}$  belong to  $S$ , Lemma 2.1 guarantees that if  $z = (p, q)$  and  $z' = (p', q')$  belong to  $Z$ , then

$$\xi_z = \xi_{z'} \quad \text{and} \quad \alpha(z) = \alpha(z')$$

if  $p = p'$  and

$$\xi_z = \xi_{z'} \quad \text{and} \quad \beta(z) = \beta(z')$$

if  $q = q'$ , so we conclude that there exist  $\xi$  in  $S$ , conull Borel sets

$$X \subset G/N \quad \text{and} \quad Y \subset X \times X$$

and maps  $\alpha_o, \beta_o : X \rightarrow \mathbb{T}$  such that

$$\xi_z = \xi \quad \text{and} \quad \alpha(z) = \alpha_o(p) \quad \text{and} \quad \beta(z) = \beta_o(q)$$

for all  $z = (p, q)$  in  $Y$ .

Hence,

$$p^{-1}A \cap N \subset \xi^{-1}(I_o \alpha_o(p)) \quad \text{and} \quad Bq^{-1} \cap N \subset \xi^{-1}(J_o \beta_o(p))$$

and

$$p^{-1}ABq^{-1} \cap N \sim \xi^{-1}(I_o J_o \alpha_o(p) \beta_o(q))$$

for all  $(p, q) \in Y$ . By Lemma 2.3, we conclude that the maps  $\alpha_o$  and  $\beta_o$  are Borel measurable on  $X$ .

**Step 2** (Almost conjugacy invariance of  $\xi$ ). We can rewrite the last identities on the form

$$A \cap Np \subset (\xi \circ c_{p^{-1}})^{-1}(I_o \alpha_o(p))p \quad \text{and} \quad B \cap Nq \subset \xi^{-1}(J_o \beta_o(q))q$$

and

$$AB \cap Npq \sim (\xi \circ c_{p^{-1}})^{-1}(I_o J_o \alpha_o(p) \beta_o(q))pq,$$

where  $(p, q) \in Z$ , and  $c_p$  denotes the automorphism of  $N$  induced by  $p$  in  $Q$ .

By the second part of Lemma 2.2, we can find a Borel measurable map  $\epsilon : Q \rightarrow \{-1, 1\}$  such that

$$\epsilon_p \circ \xi \circ c_{p^{-1}} \in S, \quad \forall p \in Q,$$

where  $\epsilon_p = \epsilon(c_{p^{-1}})$  and since  $I_o$  and  $J_o$  are assumed to be symmetric intervals, we have

$$A \cap Np \subset (\epsilon \circ \xi \circ c_{p^{-1}})^{-1}(I_o \alpha_o(p)^{\epsilon_p})p \quad \text{and} \quad B \cap Nq \subset \xi^{-1}(J_o \beta_o(q))q$$

and

$$AB \cap Npq \sim (\epsilon_p \circ \xi \circ c_{p^{-1}})^{-1}(I_o J_o (\alpha_o(p) \beta_o(q))^{\epsilon_p})pq,$$

for all  $(p, q) \in Y$ . Since

$$\epsilon_p \circ \xi \circ c_{p^{-1}} \in S$$

for almost all  $p$ , we conclude by Lemma 2.1 that

$$\epsilon_p \circ \xi \circ c_{p^{-1}} = \epsilon_{p'} \circ \xi \circ c_{p'^{-1}}$$

and

$$(\alpha_o(p) \beta_o(q))^{\epsilon_p} = (\alpha_o(p') \beta_o(q'))^{\epsilon_{p'}}$$

for all  $(p, q)$  and  $(p', q')$  in  $Y$  such that  $pq = p'q'$ . The first of these equations clearly implies that  $\xi \circ c_{p^{-1}}$  is either  $\xi$  or  $\tilde{\xi}$  for almost every (and hence all)  $p$  in  $Q$ . In particular,

$$\epsilon_p \circ \xi \circ c_{p^{-1}} = \xi, \quad \forall p \in Q,$$

which readily implies that the map  $p \mapsto \epsilon_p$  is a Borel measurable (hence continuous) homomorphism, and this finishes the proof of Proposition 1.1.



## 3. APPENDIX I: RELATIVIZATION OF CRITICAL PAIRS

The aim of this appendix is to establish the principle of relativization of critical pairs formulated in the introduction of this paper.

Let  $G$  be a compact and second countable group. Recall that the *identity component*  $G^\circ$  of  $G$  is defined as the intersection of all open subgroups of  $G$ . Since  $G$  is assumed to be compact and second countable, we can find a decreasing *sequence*  $(U_n)$  of open and *normal* subgroups such that

$$G^\circ = \bigcap_n U_n.$$

Clearly,  $G^\circ$  is a closed (but not necessarily open) normal subgroup of  $G$ .

Let  $m_G$  and  $m_{G^\circ}$  denote the Haar probability measures on  $G$  and  $G^\circ$  and note that we may regard  $m_{G^\circ}$  as a Borel probability measure on  $G$  supported on the subgroup  $G^\circ$ . By uniqueness of Haar probability measures on compact groups, we have

$$m_G = \int_G (r_x)_* m_{G^\circ} dm_G(x),$$

where  $r_x : G \rightarrow G$  denotes the right multiplication on  $G$  by the element  $x$  in  $G$ . We shall also write

$$(m_G)_x^{G^\circ} = (r_x)_* m_{G^\circ}$$

for  $x$  in  $G$ .

Recall that a pair  $(A, B)$  of Borel sets in  $G$  is *critical with respect to*  $G^\circ$  if there exist conull Borel sets

$$X \subset G \quad \text{and} \quad Z \subset X \times X$$

such that

$$m_G(A) = (m_G)_x^{G^\circ}(A) \quad \text{and} \quad m_G(B) = (m_G)_x^{G^\circ}(B)$$

for all  $x, y \in X$  and

$$(m_G)_{xy}^{G^\circ}(AB) = (m_G)_x^{G^\circ}(A) + (m_G)_y^{G^\circ}(B)$$

for all  $(x, y)$  in  $Z$ . Note that if a pair  $(A, B)$  is critical with respect to  $G^\circ$ , then in particular it is critical.

More generally, if  $N < G$  is any closed subgroup of  $G$ , we can define

$$(m_G)_x^N = (r_x)_* m_N, \quad x \in G,$$

and we still have

$$m_G = \int_G (r_x)_* m_N dm_G(x).$$

There is an obvious extension of the notion of a critical pair with respect to  $N$  defined above. In the important special case when  $N$  is an *open and normal* subgroup of  $G$ , this notion can be described as follows. There exist conull Borel sets

$$X \subset G \quad \text{and} \quad Z \subset G \times G$$

such that

$$m_G(A) = \frac{m_G(A \cap Ux)}{m_G(U)} \quad \text{and} \quad m_G(B) = \frac{m_G(B \cap Uy)}{m_G(U)}$$

for all  $x, y \in X$  and

$$m_G(AB \cap xUy) = m_G(A \cap xU) + m_G(B \cap Uy)$$

for all  $(x, y) \in Z$ .

Let  $U$  be an open and normal subgroup of  $G$ . A pair  $(A, B)$  of Borel sets in  $G$  is said to be *almost sub-critical in  $U$* , if there exist conull Borel sets

$$A' \subset A \quad \text{and} \quad B' \subset B$$

and  $p$  and  $q$  in  $G$  such that the pair

$$(p^{-1}A' \cap U, B'q^{-1} \cap U)$$

is sub-critical. If  $(A, B)$  is almost sub-critical with respect to *some* open and normal subgroup of  $G$ , then we shall simply say that  $(A, B)$  is *locally sub-critical*.

The main result of this appendix can now be formulated as follows.

**Proposition 3.1** (Relativization of critical pairs). *Suppose  $(A, B)$  is a critical pair in  $G$  which is not locally sub-critical. Then there exist an open subgroup  $L < G$  and conull  $\sigma$ -compact sets*

$$A' \subset A \quad \text{and} \quad B' \subset B$$

and  $s, t \in G$  such that

$$sA' \subset L \quad \text{and} \quad B't \subset L$$

and  $(sA', B't)$  is critical in  $L$  with respect to the identity component of  $L$ .

The proof of this proposition will be an immediate consequence of the following two propositions, which we now describe.

Given an open and normal subgroup  $U < G$  and a Haar measurable set  $C \subset G$ , we define

$$C_x = C \cap Ux \quad \text{and} \quad C^+ = \{y \in G/U : C_y \neq \emptyset\} \subset G/U,$$

for  $x$  in  $G$ . We shall say that a  $C \subset G$  is  *$U$ -balanced* if

$$e \in C^+ \quad \text{and} \quad m_G(C_y) > 0, \quad \forall y \in C^+.$$

Clearly, if  $C$  is  $U$ -balanced, then so is every conull subset of  $C$ . Finally, we say that a pair  $(A, B)$  is  *$U$ -balanced* if both  $A$  and  $B$  are  $U$ -balanced sets.

**Proposition 3.2.** *Let  $U$  be an open normal subgroup of  $G$  and suppose that  $(A, B)$  is a  $U$ -balanced critical pair in  $G$ , which is not almost sub-critical in  $U$ . Then  $(A, B)$  is critical with respect to  $U$ .*

Let  $(U_n)$  be a decreasing sequence of open and normal subgroups of  $G$  whose intersection equals the identity component of  $G$  and suppose  $(A, B)$  is a critical pair in  $G$ . For every  $n$ , we define

$$A_n = A \cap A_n^+ U_n \quad \text{and} \quad B_n = B \cap B_n^+ U_n$$

and note that

$$m_G(A_n) = m_G(A) \quad \text{and} \quad m_G(B_n) = m_G(B)$$

for all  $n$ . Hence, the intersections

$$A' = \bigcap_n A_n \quad \text{and} \quad B' = \bigcap_n B_n$$

are conull subsets of  $A$  and  $B$  respectively, and by construction, they satisfy

$$m_G(A' \cap U_n x) > 0 \quad \text{and} \quad m_G(B' \cap U_n y) > 0$$

for all  $x \in A'^+$  and  $y \in B'^+$ . Upon passing to conull  $\sigma$ -compact subsets and translating, we see that for every pair  $(A, B)$  of Haar measurable sets, there exist conull  $\sigma$ -compact subsets  $A' \subset A$  and  $B' \subset B$  and  $s, t \in G$  such that  $sA'$  and  $B't$  are  $U$ -balanced sets, and

$$m_G(A'B') \leq m_G(A') + m_G(B').$$

If the inequality would be strict, then  $(A', B')$  is sub-critical and thus  $(A, B)$  would be locally sub-critical.

We conclude that if  $(A, B)$  is a critical pair in  $G$  which is not locally sub-critical, then there exist conull  $\sigma$ -compact sets

$$A' \subset A \quad \text{and} \quad B' \subset B$$

and  $s, t \in G$  such that  $(sA', B't)$  is  $U_n$ -balanced for every  $n$  and thus, by Proposition 3.2, critical with respect to  $U_n$  for every  $n$ . Proposition 3.1 now follows immediately from the following result.

**Proposition 3.3.** *Let  $G$  be a compact group and suppose that  $(N_k)$  is a decreasing sequence of closed normal subgroups of  $G$  with intersection  $N$ . If  $(A, B)$  is a pair of Borel sets in  $G$  which is critical with respect to  $N_k$  for every  $k$ , then  $(A, B)$  is critical with respect to  $N$ .*

**3.1. Proof of Proposition 3.2.** Recall that if  $U < G$  is an open subgroup of  $G$  and  $C \subset G$  is a Borel set, we define

$$C^+ = \left\{ x \in G/U : C \cap Ux \neq \emptyset \right\} \subset G/U.$$

Proposition 3.3 is now contained in the following result.

**Proposition 3.4.** *Let  $U$  be an open normal subgroup of  $G$  and suppose  $(A, B)$  is a  $U$ -balanced pair of Borel sets in  $G$ . Fix  $x_0 \in A^+$  such that*

$$m_G(A_{x_0}) = \max_{x \in A^+} m_G(A_x).$$

*If  $(A, B)$  is not almost sub-critical in  $U$  and there exists  $z_0 \in (AB)^+ \setminus x_0 B^+$ , then*

$$m_G(AB) \geq m_G(A_x B_y) + \frac{|B^+|}{|A^+|} \cdot m_G(A) + m_G(B),$$

*for all  $x \in A^+$  and  $y \in B^+$  such that  $xy = z_0$ . Furthermore, if the pair  $(A, B)$  in addition is critical, then*

$$A^+ = B^+$$

*and  $A^+$  is a subgroup of  $G/U$  and  $(A, B)$  is a critical pair in the compact group  $L = A^+U$  with respect to closed normal subgroup  $U$  of  $L$ .*

*Proof.* Since  $(A, B)$  is not almost sub-critical in  $\mathcal{U}$ , we have

$$m_G(A_x B_y) \geq m_G(A_x) + m_G(B_y) \quad \forall (x, y) \in A^+ \times B^+,$$

and since  $\mathcal{U}$  is normal, we have

$$AB = \bigsqcup_{z \in (AB)^+} AB \cap \mathcal{U}z = \bigsqcup_{z \in (AB)^+} \left( \bigcup_{xy=z} A_x B_y \right).$$

Pick  $x_o$  in  $A^+$  which satisfies

$$m_G(A_{x_o}) = \max_{x \in A^+} m_G(A_x)$$

and note that  $m_G(A) \leq |A^+| \cdot m_G(A_{x_o})$ . In particular, if  $(A, B)$  is critical and not locally sub-critical, then

$$\begin{aligned} |A^+| \cdot m_G(A_{x_o}) + m_G(B) &\geq m_G(A) + m_G(B) = m_G(AB) \\ &\geq m_G \left( \bigcup_{y \in B^+} A_{x_o} B_y \right) \\ &= \sum_{y \in B^+} m_G(A_{x_o} B_y) \\ &\geq \sum_{y \in B^+} m_G(A_{x_o}) + m_G(B_y) \\ &\geq |B^+| \cdot m_G(A_{x_o}) + m_G(B), \end{aligned}$$

from which we can conclude that  $|A^+| \geq |B^+|$ . Arguing symmetrically with  $y_o$  in  $B^+$  which satisfies

$$m_G(B_{y_o}) = \max_{y \in B^+} m_G(B_y),$$

we conclude that  $|A^+| = |B^+|$  if  $(A, B)$  is critical and not locally sub-critical.

Suppose there exists  $z_o \in (AB)^+ \setminus x_o B^+$ . Then, for all  $x \in A^+$  and  $y \in B^+$  with  $xy = z_o$ , we have

$$\begin{aligned} m_G(AB) &= \sum_{z \in (AB)^+} m_G \left( \bigcup_{xy=z} A_x B_y \right) \\ &\geq m_G(A_{x_o} B_{y_o}) + \sum_{z \in x_o B^+} m_G(A_{x_o} B_{x_o^{-1}z}) \\ &\geq m_G(A_{x_o} B_{y_o}) + \sum_{z \in x_o B^+} m_G(A_{x_o}) + m_G(B_{x_o^{-1}z}) \\ &\geq m_G(A_{x_o} B_{y_o}) + |B^+| \cdot m_G(A_{x_o}) + m_G(B) \\ &\geq m_G(A_{x_o} B_{y_o}) + \frac{|B^+|}{|A^+|} \cdot m_G(A) + m_G(B) \\ &> \frac{|B^+|}{|A^+|} \cdot m_G(A) + m_G(B), \end{aligned}$$

since  $m_G(A_x)$  and  $m_G(B_y)$  are both positive. In particular, if  $(A, B)$  is a critical, and thus  $|A^+| = |B^+|$ , we see that the above chain of inequalities is impossible, so we must conclude that

$$(AB)^+ = A^+ B^+ = x_o B^+.$$

Since  $A$  and  $B$  are  $U$ -balanced, and thus  $e \in A^+ \cap B^+$ , we have

$$B^+ = x_o^{-1} A^+ B^+ \supset x_o^{-1} A^+ \quad \text{and} \quad x_o^{-1} A^+ = B^+ \quad \text{and} \quad B^+ B^+ = B^+,$$

since  $|A^+| = |B^+|$ . Put differently,  $B^+$  is a semigroup of the *finite* group  $G/U$ . Since any semigroup of a finite group is in fact a subgroup, this shows that  $A^+ = B^+$  and thus, by a closer study of the series of inequalities above (and arguing symmetrically with the roles of  $A$  and  $B$  and  $x_o$  and  $y_o$  interchanged), we conclude that

$$m_G(A \cap Ux) = \frac{m_G(A)}{|A^+|} \quad \text{and} \quad m_G(B \cap Uy) = \frac{m_G(B)}{|B^+|}$$

and

$$m_G(AB \cap Uxy) = m_G(A \cap Ux) + m_G(B \cap Uy)$$

for all  $x, y \in L = A^+ U$ . Hence  $(A, B)$  is a critical pair in  $L$  with respect to  $U$  (note that this does not mean that  $(A, B)$  is a critical pair in  $G$  with respect to  $U$ , unless of course  $A^+ = G/U$ ).  $\square$

**3.2. Proof of Proposition 3.3.** The aim of this subsection is to show that criticality with respect to a decreasing sequence of closed normal subgroups is inherited by the intersection of these subgroups. This will follow from rather standard differentiation lemmata which we prove here for completeness.

Recall that if  $G$  is a compact group and  $x \in G$ , we denote by  $r_x : G \rightarrow G$  the right multiplication in  $G$  by  $x$ . Clearly, this map induces a continuous selfmap on the space of probability measures on  $G$  equipped with the weak\*-topology.

**Proposition 3.5.** *Suppose  $(\mu_n)$  is a sequence of Borel probability measures on  $G$  with weak\*-limit  $\mu$ . For every Borel set  $B \subset G$  and subsequence  $(n_k)$ , there exist a further subsequence  $(n_{k_j})$  and a Haar conull set  $X \subset G$  such that*

$$\lim_{j \rightarrow \infty} (r_x)_* \mu_{n_{k_j}}(B) = (r_x)_* \mu(B)$$

for all  $x \in X$ .

*Proof.* Define the functions

$$\psi_n(x) = (r_x)_* \mu_n(B) \quad \text{and} \quad \psi(x) = (r_x)_* \mu(B), \quad \text{and} \quad x \in G.$$

By Theorem 444F(e) in [3], these functions are Borel measurable on  $G$ , so it suffices to show (and then pass to an almost everywhere convergent subsequence) that  $\psi_n \rightarrow \psi$  in  $L^2(G, m_G)$  (in the norm topology), or equivalently, that  $\psi_n \rightarrow \psi$  (in the weak topology) and

$$\lim_{n \rightarrow \infty} \int_G |\psi_n(x)|^2 dm_G(x) = \int_G |\psi(x)|^2 dm_G(x).$$

Fix  $\varphi$  in  $L^2(X, m_G)$  and note that

$$\int_G \varphi(x) \psi_n(x) dm_G(x) = \int_G \left( \int_G \varphi(x) \chi_B(yx^{-1}) dm_G(x) \right) d\mu_n(y)$$

by Fubini's Theorem, and the function

$$\eta_\varphi(y) = \int_G \varphi(x) \chi_B(yx^{-1}) \, dm_G(x), \quad y \in G,$$

is continuous on  $G$ . Since  $\mu_n \rightarrow \mu$  in the weak\*-topology, we have

$$\int_G \varphi(x) \psi_n(x) \, dm_G(x) = \int_G \eta_\varphi \, d\mu_n \rightarrow \int_G \eta_\varphi \, d\mu = \int_G \varphi(x) \psi(x) \, dm_G(x).$$

Since  $\varphi$  is arbitrary, we conclude that  $\psi_n \rightarrow \psi$  weakly in  $L^2(G, m_G)$ .

The proof that the norms converge as well follows very similar lines as the proof of the weak convergence and is omitted.  $\square$

**Corollary 3.6.** *Suppose  $(N_k)$  is a decreasing sequence of closed subgroups of  $G$  with intersection  $N$ . For every Borel set  $B \subset G$  and subsequence  $(n_k)$ , there exist a further subsequence  $(n_{k_j})$  and a Haar conull subset  $X \subset G$  such that*

$$\lim_{j \rightarrow \infty} (m_G)_x^{N_{k_j}}(B) = (m_G)_x^N(B)$$

for all  $x$  in  $G$ .

*Proof.* Recall that if  $M$  is a closed subgroup of  $G$ , then

$$(m_G)_x^M = (r_x)_* m_M, \quad \forall x \in G,$$

where  $m_M$  is considered as a Borel probability measure on  $G$ . Since  $m_{N_k} \rightarrow m_N$  in the weak\*-topology on  $G$ , the corollary follows immediately from Proposition 3.5.  $\square$

Recall that if  $B \subset G$  is a Borel set, then the *essential stabilizer* of  $B$  is the closed subgroup

$$E_B = \{g \in G : m_G(B \Delta gB) = 0\},$$

where  $\Delta$  denotes the symmetric set difference.

**Corollary 3.7.** *Let  $G$  be a compact and second countable group and suppose  $(U_n)$  is a countable basis for the topology of  $G$ . If  $A, B \subset G$  are pre-compact Borel sets such that*

$$m_G(A \cap U_n) = m_G(B \cap U_n)$$

for all  $n$ , then  $A \sim B$ . In particular, if  $C \subset G$  is a pre-compact Borel set and there exist  $x, y \in G$  such that

$$m_G(Cx \cap U_n) = m_G(Cy \cap U_n)$$

for all  $n$ , then  $xy^{-1}$  belongs to the essential stabilizer of the set  $C$ .

*Proof.* Let  $(V_n)$  be a decreasing sequence of open neighborhoods of the identity in  $G$ . The assumption above in particular implies that

$$\frac{m_G(Ax^{-1} \cap V_n)}{m_G(V_n)} = \frac{m_G(Bx^{-1} \cap V_n)}{m_G(V_n)}, \quad \forall x \in G,$$

for all  $n$ , or equivalently, that  $(r_x)_* \mu_n(A) = (r_x)_* \mu_n(B)$  for all  $n$  and for all  $x$  in  $G$ , where

$$\mu_n(\cdot) = \frac{m_G(\cdot \cap V_n)}{m_G(V_n)}.$$

One readily shows that  $\mu_n \rightarrow \delta_e$  in the weak\*-topology on the space of probability measures on  $G$  and thus  $\chi_A(x) = \chi_B(x)$  for Haar almost every  $x$  in  $G$  by Proposition 3.5, which finishes the proof.  $\square$

3.2.1. *Proof of Proposition 3.3.* Let  $G$  be a compact group and let  $(N_k)$  be a decreasing sequence of closed normal subgroups in  $G$  with intersection  $N$ . Suppose  $(A, B)$  is a pair of Borel sets in  $G$ , which is critical with respect to  $N_k$  for every  $k$ , i.e. there exist Borel sets

$$X_k \subset G \quad \text{and} \quad Z_k \subset X_k \times X_k$$

of full Haar measures such that

$$m_G(A) = (m_G)_x^{N_k}(A) \quad \text{and} \quad m_G(B) = (m_G)_y^{N_k}(B)$$

for all  $x, y \in X_k$  and

$$(m_G)_{xy}^{N_k}(AB) = (m_G)_x^{N_k}(A) + (m_G)_y^{N_k}(B)$$

for all  $(x, y)$  in  $Z_k$ . By Corollary 3.6, there exist a Haar conull set  $X' \subset G$  and a subsequence  $(k_j)$  such that

$$(m_G)_x^N(A) = \lim_{j \rightarrow \infty} (m_G)_x^{N_{k_j}}(A) \quad \text{and} \quad (m_G)_y^N(B) = \lim_{j \rightarrow \infty} (m_G)_y^{N_{k_j}}(B)$$

and

$$(m_G)_z^N(AB) = \lim_{j \rightarrow \infty} (m_G)_z^{N_{k_j}}(AB)$$

for all  $x, y, z \in X'$ . In particular,

$$m_G(A) = (m_G)_x^N(A) \quad \text{and} \quad m_G(B) = (m_G)_y^N(B)$$

and

$$(m_G)_{xy}^N(AB) = (m_G)_x^N(A) + (m_G)_y^N(B)$$

for all  $(x, y)$  in  $Z$ , where

$$X = X' \cap \left( \bigcap_k X_k \right) \quad \text{and} \quad Z = p^{-1}(X') \cap \left( \bigcap_k Z_k \right)$$

and  $p : G \times G \rightarrow G$  denotes the product map. This shows that  $(A, B)$  is critical with respect to  $N$ .

#### 4. APPENDIX II: RIGIDITY OF CRITICAL PAIRS

The aim of this appendix is to establish the rigidity of critical pairs formulated in the introduction, namely the principle (which was essentially pointed out already by Kneser in [10]) that critical pairs which are sufficiently "regular" must be "maximal", in the sense that any other critical pair which almost reduces to it, must in fact reduce to it.

Let  $G$  and  $M$  be compact groups. Recall that a closed set  $I \subset M$  is *regular* if it equals the closure of its interior, and we say that a pair  $(I, J)$  of Borel sets in  $M$  is *left stable* if the inclusion  $xJ \subset IJ$  implies that  $x \in I$ . There is an obvious analogous notion of a right stable pair, and we say that  $(I, J)$  is *stable* if it is both left and right stable.

Suppose  $(I, J)$  is a stable pair in  $M$  and there exists a compact subgroup  $K < \text{Aut}(M)$  which fixes  $I$  and  $J$ . Then it is not hard to show that

$$(I \rtimes K, J \rtimes K)$$

is stable in the group  $M \rtimes K$ , and if  $\pi : G \rightarrow M \rtimes K$  is a continuous surjective homomorphism, then

$$(p^{-1}(I \rtimes K), p^{-1}(J \rtimes K))$$

is stable in  $G$ . Finally, since  $\pi$  is necessarily open, we see that if  $I$  and  $J$  are regular sets, then so are the sets

$$p^{-1}(I \rtimes K) \quad \text{and} \quad p^{-1}(J \rtimes K).$$

In particular, the principle of rigidity of critical pairs mentioned in the introduction follows immediately from the following result.

**Proposition 4.1.** *Suppose  $(A_1, B_1)$  and  $(A_2, B_2)$  are critical pairs in  $G$  such that*

$$A_1 \sim A_2 \quad \text{and} \quad B_1 \sim B_2.$$

*If  $(A_2, B_2)$  is stable and  $A_2$  and  $B_2$  are regular sets, then*

$$A_1 \subset A_2 \quad \text{and} \quad B_1 \subset B_2.$$

*Proof.* Define the sets

$$A_o = A_1 \cap A_2 \quad \text{and} \quad B_o = B_1 \cap B_2$$

and note that

$$m_G(A_o) = m_G(A_1) = m_G(A_2) \quad \text{and} \quad m_G(B_o) = m_G(B_1) = m_G(B_2),$$

and

$$\begin{aligned} m_G(A_o B_o) &\leq m_G(A_1 B_1) \\ &= m_G(A_1) + m_G(B_1) \\ &= m_G(A_o) + m_G(B_o). \end{aligned}$$

If the first inequality would be strict, then  $(A_o, B_o)$  is sub-critical, and thus, by Kemperman's Theorem, there exists an open subgroup of  $G$  such that  $A_o B_o = A_o \cup B_o$ . Define

$$V_1 = A_2^o \setminus A_o \cup \quad \text{and} \quad V_2 = B_2^o \setminus \cup B_o.$$

Since  $A_o \cup$  and  $\cup B_o$  are clopen in  $G$  and

$$m_G(A_2^o) = m_G(A_2) \quad \text{and} \quad m_G(B_2^o) = m_G(B_2),$$

we see that  $V_1$  and  $V_2$  are open null sets and thus empty. Since  $A_2$  and  $B_2$  are regular sets, we conclude that

$$A_2 = \overline{A_2^o} \subset A_o \cup \quad \text{and} \quad B_2 = \overline{B_2^o} \subset \cup B_o,$$

and

$$A_2 B_2 \subset A_o \cup B_o = A_o B_o \subset A_2 B_2.$$

In particular, we see that  $(A_o, B_o)$  cannot be sub-critical, and so we can henceforth assume that

$$m_G(A_o B_o) = m_G(A_1 B_1) = m_G(A_2 B_2).$$



We wish to prove that  $A_1 \subset A_2$  and  $B_1 \subset B_2$ . Assume for the sake of contradiction that there exists an element  $x \in A_1 \setminus A_2$ . Then,

$$\begin{aligned}
 m_G(A_1) + m_G(B_1) &= m_G(A_1 B_1) \\
 &\geq m_G((A_o \cup \{x\})B_o) \\
 &= m_G((A_o B_o \cup (xB_o \setminus A_o B_o))) \\
 &= m_G(A_o B_o) + m_G(xB_o \setminus A_o B_o) \\
 &= m_G(A_o) + m_G(B_o) + m_G(xB_2 \setminus A_2 B_2) \\
 &= m_G(A_1) + m_G(B_1) + m_G(xB_2 \setminus A_2 B_2),
 \end{aligned}$$

which forces the set  $xB_2 \setminus A_2 B_2$  to be a null set. Since  $A_2$  and  $B_2$  are closed sets, so is  $A_2 B_2$ . In particular, the set  $xB_2^o \setminus A_2 B_2$  is open and null, which forces it to be empty. Since  $B_2$  is regular, we have

$$x\overline{B_2^o} = xB_2 \subset A_2 B_2.$$

By assumption,  $(A_2, B_2)$  is left stable, and thus  $x \in A_2$ , which is a contradiction. This shows that  $A_1 \subset A_2$ . The proof that  $B_1 \subset B_2$  is completely analogous and we omit it.  $\square$

### 5. APPENDIX III: MOTIVATION AND APPLICATIONS

In this appendix we outline the motivation for the investigations undertaken in this paper and describe some applications of the main result. A significant part of the material in this section has been developed in collaboration with A. Fish.

A classical and very useful fact, which is often attributed to Steinhaus, asserts that the product of any two Borel measurable sets in a (locally) compact group with positive Haar measures has non-empty interior. Many early results on small product sets in compact groups can be viewed as attempts to quantify Steinhaus observation. For instance, Kneser's Theorem can be restated along these lines as follows. Let  $G$  be a compact and connected abelian group and suppose  $B, C \subset G$  are Borel measurable subsets with positive Haar measures such that

$$m_G(CB^{-1}) = m_G(C) + m_G(B) < 1.$$

Then the product set  $CB^{-1}$  contains a conull subset which is the pull-back of an open interval in  $\mathbb{T}$  under a continuous surjective homomorphism  $\xi : G \rightarrow \mathbb{T}$ . Indeed, let  $A = (CB^{-1})^c$  and note that

$$m_G(AB) = m_G(A) + m_G(B) < 1.$$

By Kneser's Theorem, there exists a continuous surjective homomorphism  $\xi : G \rightarrow \mathbb{T}$  and closed intervals  $I$  and  $J$  of  $\mathbb{T}$  such that  $A$  and  $B$  are conull subsets of the pull-backs  $\xi^{-1}(I)$  and  $\xi^{-1}(J)$  respectively, and thus

$$A^c = CB^{-1} \supset \xi^{-1}(I^c).$$

After the appearance of Steinhaus result in [12], many attempts were made to establish natural analogs of the phenomenon in discrete groups. One of the earliest succesful attempts in this direction was made by Følner in [4]. To explain his result, we need to introduce some notation.

Recall that a countable group is *amenable* if there exists a left  $G$ -invariant positive and normalized functional  $\lambda$  on  $\ell(G)$ . Note that  $\lambda$  gives rise to a left  $G$ -invariant finitely additive probability measure  $\lambda'$  on  $G$  by

$$\lambda'(C) = \lambda(\chi_C), \quad C \subset G.$$

Let  $\mathcal{L}_G$  denote the set of all left invariant means on  $G$  and note that  $\mathcal{L}_G$  is a weak\*-compact convex subset of  $\ell^\infty(G)^*$ . We stress that  $\mathcal{L}_G$  is always a very large set if  $G$  is infinite; the cardinality of its set of extremal points is equal to the power set of the continuum. If  $\mathcal{C} \subset \mathcal{L}_G$ , then we define the *upper* and *lower  $\mathcal{C}$ -density* of a subset  $C \subset G$  by

$$d_{\mathcal{C}}^*(C) = \sup_{\lambda \in \mathcal{C}} \lambda'(C) \quad \text{and} \quad d_{\mathcal{C}}^{\mathcal{C}}(C) = \inf_{\lambda \in \mathcal{C}} \lambda'(C),$$

respectively. In particular, if  $\mathcal{C} = \mathcal{L}_G$ , then we shall simply write

$$d^* = d_{\mathcal{L}_G}^* \quad \text{and} \quad d_* = d_{\mathcal{L}_G}^{\mathcal{L}_G}.$$

In [4], Følner proves that if  $B$  and  $C$  are subsets of  $G$  with  $d^*(B) > 0$  and  $d_*(C) > 0$ , then there exists a compact group  $K$ , a homomorphism  $\tau : G \rightarrow K$  with dense image, an open set  $U \subset K$  and a set  $T \subset G$  with  $d_*(T) = 1$  such that

$$CB^{-1} \supset \tau^{-1}(U) \cap T.$$

One of the main open problems in this area of research is whether the set  $T$  above can be dispensed with, that is to say, whether the containment of the *Bohr set*  $\tau^{-1}(U)$  in  $CB^{-1}$  is *global*.

Around the same time as Følner's Theorem came out, Kneser [10] proved that if  $CB^{-1}$  is sufficiently small, then such a global containment is indeed possible to deduce. More precisely, Kneser considered the set  $\mathcal{B} \subset \mathcal{L}_{\mathbb{Z}}$  of *Birkhoff means* on the additive group  $\mathbb{Z}$  of integers, which consists of all accumulation points of the sequence

$$\lambda_n(\varphi) = \frac{1}{n} \sum_{k=1}^n \varphi(k), \quad \varphi \in \ell^\infty(\mathbb{Z})$$

in the weak\*-topology on  $\ell^\infty(\mathbb{Z})$  (one readily checks that these means are always invariant) and established the following result.

**Kneser's Density Theorem.** *Suppose  $A, B \subset \mathbb{N}$  satisfy*

$$d_*^{\mathcal{B}}(AB) < d_*^{\mathcal{B}}(A) + d_*^{\mathcal{B}}(B).$$

*Then  $AB$  is a cofinite subset of periodic set in  $\mathbb{N}$ .*

In particular, if we set  $C = (AB)^c$ , then  $C \subset \mathbb{N}$  and

$$d_{\mathcal{B}}^*(CB^{-1}) < \min(1, d_{\mathcal{B}}^*(C) + d_*^{\mathcal{B}}(B)),$$

and conversely, if this inequality holds, then

$$d_*^{\mathcal{B}}(AB) < d_*^{\mathcal{B}}(A) + d_*^{\mathcal{B}}(B)$$

with  $A = (CB^{-1})^c$ . We conclude that if  $(C, B)$  satisfies the bound above, then  $CB^{-1}$  contains a periodic subset of the integers, that is to say, a subset of the form  $\tau^{-1}(U)$ , where  $\tau : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  is the canonical map and  $U \subset \mathbb{Z}/n\mathbb{Z}$  is non-empty. In particular,  $CB^{-1}$  globally contains a Bohr

set.

In a recent joint paper [1] with A. Fish, the author develops a general technique to establish global Bohr containment results for products of two subsets  $C'$  and  $B'$  of a countable amenable group  $G$  which satisfy the much weaker bound

$$d_*(C'B'^{-1}) \leq \min(1, d_*(C') + d^*(B')).$$

Note that if we set  $A' = (C'B'^{-1})^c$ , then

$$d_*(A'^{-1}C') \leq \min(1, d^*(A') + d_*(C')).$$

In order to explain the relevance of Theorem 1.1 and Corollary 1.2 to our approach, we shall formulate a weak version of one of the technical innovations in our paper, which we refer to as a *correspondence principle for product sets*.

The idea is that one can associate, to every pair of subsets  $(A', C')$  of a countable amenable group  $G$ , a compact and second countable group  $K$ , a continuous homomorphism  $\tau : G \rightarrow K$  with dense image and Borel sets  $\tilde{A}, \tilde{C}$  such that  $A$  and  $C$  are *essentially contained* in pull-backs of the sets  $\tilde{A}$  and  $\tilde{C}$  under the map  $\tau$ . The exact formulation is quite technical and will be given below.

However, before we state our Correspondence Principle, we wish to point out an important feature of countable amenable groups, which should explain our choice to concentrate on products in compact groups which satisfy the assumptions of Corollary 1.2.

**Proposition 5.1.** *Let  $G$  be a compact group which contains a dense countable amenable subgroup. Then the identity component of  $G$  is abelian.*

*Proof.* The proof combines a series of classical observations. A good source to the material used in the proof is the book [7].

Firstly, recall that by Peter-Weyl's Theorem, there exist a decreasing net  $(N_\alpha)$  of closed subgroups in  $G$  with trivial intersection such that  $G/N_\alpha$  is a Lie group for every  $\alpha$ , i.e. it sits a closed subgroup inside a unitary group  $U(n)$  for some  $n$ . In particular, the identity component  $G_\alpha$  of the quotient group  $G/N_\alpha$  has finite index in the  $G/N_\alpha$ .

Now, if  $\Gamma$  is a dense countable amenable subgroup of  $G$ , then, for every  $\alpha$ , there exists a finite index subgroup  $\Gamma_\alpha$  of  $\Gamma$  which projects onto a dense countable amenable subgroup of  $G_\alpha$ . Let  $R_\alpha$  denote the solvable radical of  $G_\alpha$ . Then  $S_\alpha = G_\alpha/R_\alpha$  is a connected semisimple Lie group which contains a dense countable amenable subgroup, namely the projection of  $\Gamma_\alpha$  onto its image in  $S_\alpha$ .

If  $S_\alpha$  is non-trivial, then, by Tits' alternative, the last dense subgroup must contain a free group on two generators which would contradict its amenability. We conclude that  $G_\alpha$  equals its solvable radical and is thus a compact solvable connected Lie group, and hence abelian (by the structure theory of such groups).

We have shown that  $G_\alpha$  is abelian for every  $\alpha$ . If we denote by  $V_\alpha$  the pull-back of  $G_\alpha$  in  $G$  under the canonical map  $p_\alpha$  from  $G$  onto  $G/N_\alpha$ , then we see that  $V_\alpha$  is an open and closed subgroup of  $G$  for every  $\alpha$ , and thus the intersection  $C$  of the family  $(V_\alpha)$  contains the identity component of  $G$ . Fix any two elements  $x$  and  $y$  in  $C$  and note that  $p_\alpha$  kills the commutator of  $x$  and  $y$  for every  $\alpha$ . Since the kernel of  $p_\alpha$  is equal to  $N_\alpha$  and the intersection of all the

family  $(N_\alpha)$  is trivial, we conclude that the commutator of  $x$  and  $y$  is trivial in  $G$ , i.e. the closed subgroup  $C$  is abelian, which forces the identity component of  $G$  to be abelian.  $\square$

We can now formulate our correspondence principle for product sets. Let  $G$  be a countable group and suppose  $X$  is a compact metrizable  $G$ -space, i.e.  $X$  is a compact and metrizable space equipped with an action of  $G$  by homeomorphisms. We say that a point  $x_o$  in  $X$  is  $G$ -transitive if its orbit  $G \cdot x_o$  is dense in  $X$ . Given any subset  $B \subset X$  and  $x$  in  $X$ , we define the set  $B_x \subset G$  by

$$B_x = \{g \in G : g \cdot x \in B\}.$$

If  $K$  is a compact group and  $\tau : G \rightarrow K$  is a homomorphism with dense image, then  $K$  is a natural  $G$ -space, where  $G$  acts on  $K$  by right translations via  $\tau$ . Note that every point in  $K$  is  $G$ -transitive.

We may also consider the diagonal action of  $G$  on the product  $X \times K$  and if  $\mu$  is a  $G$ -invariant probability measure on  $X$ , then we say a  $G$ -invariant probability  $\eta$  on  $X \times K$  is a *joining* of  $\mu$  and  $m_K$  if the natural projections of  $\eta$  onto  $X$  and  $K$  equal  $\mu$  and  $m_K$  respectively. Finally, recall that a  $G$ -invariant probability measure  $\mu$  on a compact  $G$ -space is *ergodic* if every  $G$ -invariant Borel set is either null or conull with respect to  $\mu$ .

**Correspondence Principle.** *Let  $G$  be a countable amenable group and suppose  $A', C' \subset G$ . Then there exist compact metrizable  $G$ -spaces  $X$  and  $Y$  with  $G$ -transitive points  $x_o$  and  $y_o$  respectively, clopen sets  $A \subset X$  and  $C \subset Y$  such that*

$$A' = A_{x_o} \quad \text{and} \quad C' = C_{y_o}.$$

*Furthermore, there exists ergodic  $G$ -invariant Borel probability measures  $\mu$  and  $\nu$  on  $X$  and  $Y$  respectively such that*

$$d^*(A') = \mu(A) \quad \text{and} \quad d_*(C') \leq \nu(C).$$

*Moreover there exist a compact second countable group  $K$  with an abelian identity component, a homomorphism  $\tau : G \rightarrow K$  with dense image, Borel measurable sets  $\tilde{A}, \tilde{C} \subset K$  with*

$$m_K(\tilde{A}) \geq \nu(A) \quad \text{and} \quad m_K(\tilde{C}) \geq \nu(C)$$

*and ergodic  $G$ -invariant joinings  $\eta$  on  $X \times K$  of  $\nu$  and  $m_K$  and  $\xi$  on  $Y \times K$  of  $\nu$  and  $m_K$  respectively such that*

$$A_x \subset \tau^{-1}(\tilde{A}s) \quad \text{and} \quad C_y \subset \tau^{-1}(\tilde{C}t)$$

*for almost every  $(x, s)$  and  $(y, t)$  with respect to  $\eta$  and  $\xi$  respectively and*

$$d_*(A'^{-1}C') \geq m_K(\tilde{A}^{-1}\tilde{C}).$$

In particular, if  $G$  is a countable amenable group and  $(A', C')$  is a pair of subsets of  $G$  which satisfy

$$d_*(A'^{-1}C') \leq \min(1, d^*(A') + d_*(C')),$$

then the pair  $(\tilde{A}, \tilde{C})$  above must satisfy

$$m_K(\tilde{A}^{-1}\tilde{C}) \leq \min(1, m_K(\tilde{A}) + m_K(\tilde{C})),$$

i.e. the pair  $(\tilde{A}^{-1}, \tilde{C})$  is either sub-critical or critical in  $K$ . In the first case, we can use Kemperman's Theorem to show that  $A$  must be "locally" contained in a subset of  $G$  which is invariant under a finite index subgroup and in the second case, Corollary 1.2 tells us that  $A$  either has

large "chunks" which are invariant under a finite index subgroup or it must "locally" be contained in a sturmian set.

In either case, these results are *local* in nature, and need to be complemented with further analysis in order to establish global containment. We refer the interested reader to the paper [1] for more details.

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